

On the PPN 1+2 Body Problem*

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February 7, 2008

Abstract

A particular case of the three-body problem, in the PPN formalism, is presented. The Hamiltonian function is obtained and the problem is reduced to a perturbed two-body one.

1 INTRODUCTION

Several approaches were performed on the PPN two-body problem, (Soffel, 1989). In the PPN three-body problem the most studied case is the motion of a test particle in the gravitational field of two massive bodies, i. e. the restricted problem (Brumberg, 1972). This paper presents a different situation: the motion of a binary system of two very small masses around a massive body, when the motion of the massive body not is affected by the presence of the couple. The physical situations modeled may be the motion of a binary asteroid (a binary comet fragments) or of two artificial objects. The special choice of masses (the small bodies have equal masses) particularize the last case. One uses the PPN formalism in the first approximation in the Hamiltonian formulation. In the *Jacobi* coordinates, the three-body problem splits in two coupled perturbed two-body problems: the motion of the barycenter of the binary system and the relative motion of the two small bodies. The first is rather well approximated by the motion of a test particle in the gravitational field of a massive body, e. g. Damour and Deruelle (1985), Heimberger *et al.* (1990). An imposed circular solution for this problem is substituted in the two-body relative motion. Similar to the case of the restricted three-body problem, one passes to a comoving coordinate system, where the *Ox* axis coincides with the direction given by the mass center

*GRL Report 12/04/1997

of the binary system and the center of the massive body. In this system the Hamiltonian becomes autonomous. Conclusions are given to the possibility of a GRT experiment in this framework.

We start with the usual *Lagrange* function L_N of the N-particles moving in a self-consistent field in the PPN formalism (see Soffel, 1989 for notations). We obtain the *Hamilton* function for the three-body problem by means of the *Legendre* transformations:

$$\begin{aligned}
H_3 &= \sum_i^3 \langle \mathbf{p}_i, \mathbf{v}_i \rangle - L_3 \text{ with } \mathbf{p}_i = \frac{\partial L_3}{\partial \mathbf{v}_i} \implies \\
H_3 &= \sum_i^3 \left(\frac{p_i^2}{2m_i} - \frac{p_i^4}{8c^2 m_i^3} \right) - \frac{1}{2} \sum_{i,j \neq i}^3 \frac{G m_i m_j}{r_{ij}} - \sum_{i,j \neq i}^3 \frac{G(2\gamma + 1)m_j}{2c^2 m_i r_{ij}} p_i^2 + \quad (1) \\
&+ \sum_{i,j \neq i}^3 \frac{G}{4c^2} \left(\frac{4\gamma + 3}{r_{ij}} \langle \mathbf{p}_i, \mathbf{p}_j \rangle + \frac{1}{r_{ij}^3} \langle \mathbf{p}_i, \mathbf{r}_{ij} \rangle \langle \mathbf{p}_j, \mathbf{r}_{ij} \rangle \right) + \\
&+ \frac{2\beta - 1}{2c^2} \sum_{i,j \neq i}^3 \sum_{k \neq i}^3 \frac{G^2 m_i m_j m_k}{r_{ij} r_{ik}}.
\end{aligned}$$

The Hamiltonian formalism is used in order to benefit of the *Lie-Deprit* method in the study of the near integrable systems.

2 THE PPN 1+2 BODY PROBLEM WITH EQUAL SMALL MASSES

This section describes the Hamiltonian of a particular case of the three-body problem: a close pair, $m_1 = m_2 = m$, rotating around the common center of mass and the third body, $m_3 = M$, far from these two, with $m \ll M$. We perform the symplectic transformations (*Jacobi* coordinates):

$$\mathbf{r}_1 = \frac{\mathbf{r}}{2} + \mathbf{R} + \mathbf{R}', \quad \mathbf{r}_2 = -\frac{\mathbf{r}}{2} + \mathbf{R} + \mathbf{R}', \quad \mathbf{r}_3 = \mathbf{R}', \quad (2)$$

$$\mathbf{p}_1 = \mathbf{p}_r + \frac{\mathbf{p}_R}{2}, \quad \mathbf{p}_2 = -\mathbf{p}_r + \frac{\mathbf{p}_R}{2}, \quad \mathbf{p}_3 = -\mathbf{p}_R + \mathbf{p}_{R'}.$$

Due to that \mathbf{R}' is not present in the Hamiltonian function (consequently, $\mathbf{p}_{R'}$ is a constant of motion) and considering $\mathbf{p}_{R'} = 0$, we obtain:

$$H_{1+2} = H_{Newton} + H_{PPN},$$

where

$$H_{Newton} = \frac{p_r^2}{m} - \frac{Gm^2}{r} + \frac{p_R^2(M + 2m)}{4mM} - \frac{2GmM}{R} - \quad (3)$$

$$-GmM \left(\frac{1}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|} + \frac{1}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} - \frac{2}{R} \right)$$

and

$$\begin{aligned} H_{PPN} = & -\frac{1}{8c^2 m^3} \left(2p_r^4 + \frac{p_R^4}{8} + p_R^2 p_r^2 + 2 \langle \mathbf{p}_r, \mathbf{p}_R \rangle^2 \right) - \frac{p_R^4}{8c^2 M^3} - \quad (4) \\ & -\frac{G(2\gamma+1)}{2c^2 r} \left(2p_r^2 + \frac{p_R^2}{2} \right) - \frac{G(2\gamma+1)M}{2c^2 m} \left[\left(p_r^2 + \frac{p_R^2}{4} + \langle \mathbf{p}_r, \mathbf{p}_R \rangle \right) \frac{1}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} + \right. \\ & + \left. \left(p_r^2 + \frac{p_R^2}{4} - \langle \mathbf{p}_r, \mathbf{p}_R \rangle \right) \frac{1}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|} \right] - \frac{G(2\gamma+1)mp_R^2}{2c^2 M} \left(\frac{1}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|} + \frac{1}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} \right) + \\ & + \frac{G}{2c^2} \left[\frac{4\gamma+3}{r} \left(-p_r^2 + \frac{p_R^2}{4} \right) + \frac{1}{r^3} \left(\langle \frac{\mathbf{p}_R}{2}, \mathbf{r} \rangle^2 - \langle \mathbf{p}_r, \mathbf{r} \rangle^2 \right) \right] + \\ & + \frac{G}{2c^2} \left[\frac{4\gamma+3}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} \left(-\langle \mathbf{p}_r, \mathbf{p}_R \rangle - \frac{p_R^2}{2} \right) + \frac{\langle \frac{\mathbf{p}_R}{2} + \mathbf{p}_r, \frac{\mathbf{r}}{2} + \mathbf{R} \rangle \langle -\mathbf{p}_R, \frac{\mathbf{r}}{2} + \mathbf{R} \rangle}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|^3} \right] + \\ & + \frac{G}{2c^2} \left[\frac{4\gamma+3}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|} \left(\langle \mathbf{p}_r, \mathbf{p}_R \rangle - \frac{p_R^2}{2} \right) + \frac{\langle \frac{\mathbf{p}_R}{2} - \mathbf{p}_r, \frac{\mathbf{r}}{2} - \mathbf{R} \rangle \langle -\mathbf{p}_R, \frac{\mathbf{r}}{2} - \mathbf{R} \rangle}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|^3} \right] + \\ & + \frac{(2\beta-1)G^2 m}{c^2} \left[\frac{m^2}{r^2} + \frac{mM}{r} \left(\frac{1}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|} + \frac{1}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} \right) + \right. \\ & + \left. \frac{M^2 + mM}{2} \left(\frac{1}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|} + \frac{1}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} \right)^2 - \frac{M^2}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right| \left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} \right]. \end{aligned}$$

We have

$$\frac{1}{\left| \frac{\mathbf{r}}{2} - \mathbf{R} \right|} = \sum_{n=0}^{\infty} \frac{r^n}{2^n R^{n+1}} P_n \left(\frac{\langle \mathbf{r}, \mathbf{R} \rangle}{rR} \right), \quad \frac{1}{\left| \frac{\mathbf{r}}{2} + \mathbf{R} \right|} = \sum_{n=0}^{\infty} \frac{r^n}{2^n R^{n+1}} (-1)^n P_n \left(\frac{\langle \mathbf{r}, \mathbf{R} \rangle}{rR} \right), \quad (5)$$

where P_n is the n -th usual *Legendre* polynomial.

3 THE DECOUPLING OF THE TWO PROBLEMS

A physical realization for this problem is the motion of two bodies with masses of 10^3 kg each, with 1 m relative semi-major axis (considered unperturbed) and situated at the distance of 20 mil. km. from the *Sun*. The numerical evaluations

for the perturbative terms from the Eqs. (3) and (4) will be estimated in this case. For the *Hamilton* function given by Eq. (4) the Newtonian orders of perturbations for the relative problem (m_1, m_2) are:

$$\lambda_1 = \frac{M}{2m} \left(\frac{r}{R} \right)^3 = O(10^{-4}) \quad (6)$$

and for the motion of the barycenter relative to the massive body:

$$\lambda_2 = \frac{1}{4} \left(\frac{r}{R} \right)^2 = O(10^{-21}). \quad (7)$$

In this case, the motion of the barycenter of the couple might be considered unperturbed in a good approximation, $\lambda_2 = O(\lambda_1^5)$. The problem is reduced to the study of the relative motion in the binary system. The motion of the massive body is not affected by the presence of the couple; this type of problem being considered as "1+2 body problem". By retaining the significant terms, the problem splits in two perturbed two-body problems. The motion of the barycenter is described by the following Hamiltonian:

$$H_{cm} = \frac{p_R^2(M+2m)}{4mM} - \frac{2GmM}{R} - \frac{p_R^4}{64c^2m^3} - \frac{G(2\gamma+1)M}{4c^2m} \frac{p_R^2}{R} + \quad (8)$$

$$+ \frac{(2\beta-1)G^2mM^2}{2c^2R^2} + O\left(10^{-20} \frac{p_R^2}{4m}\right).$$

The perturbing terms maintained (of the order of $O(10^{-8}p_R^2/4m)$) from Eq. (8) are given by the PPN model of the problem, which decouples from the relative problem in the binary system in the considered approximation. The resulting Hamiltonian is similar to the one for the case of the motion of a test particle in the gravitational field of a massive body. This problem admits a circular solution (Krefetz, 1967). In polar coordinates (the reference plane is that of the binary system barycenter motion)

$$R = \sqrt{X^2 + Y^2}, \quad P_R, \quad \Phi = \arctan \frac{Y}{X}, \quad P_\Phi,$$

(the canonic change of coordinates is obvious) this solution has the form:

$$R = R'_0, \quad P_R = 0, \quad P_\Phi = P'_{\Phi_0}, \quad \Phi = n't + \Phi_0. \quad (9)$$

One denotes with ' the initial conditions from the Eq. (9), in order to distinguish between them and those of the Newtonian problem, (without losing the generality, one may consider $\Phi'_0 = 0$). The relative motion in the binary system is described by the following Hamilton function:

$$H_{bin} = \frac{p_r^2}{m} - \frac{Gm^2}{r} - \frac{GmMr^2}{2R^3} P_2 \left(\frac{\langle \mathbf{r}, \mathbf{R} \rangle}{rR} \right) -$$

$$\begin{aligned}
& -\frac{1}{8c^2m^3} \left(p_R^2 p_r^2 + 2 \langle \mathbf{p}_R, \mathbf{p}_R \rangle^2 \right) + \frac{Gp_R^2}{8c^2r} - \frac{G(2\gamma+1)Mp_r^2}{c^2mR} - \\
& -\frac{G(2\gamma+1)Mr}{c^2mR^2} \langle \mathbf{p}_R, \mathbf{p}_R \rangle + \frac{G \langle \mathbf{r}, \mathbf{p}_R \rangle^2}{8c^2r^3} + \frac{2(2\beta-1)G^2m^2M}{c^2Rr} + O\left(10^{-12}\frac{p_r^2}{m}\right).
\end{aligned} \tag{10}$$

Introducing Eq. (9) in Eq. (10) one obtains:

$$\begin{aligned}
H_{bin} = & \frac{p_r^2}{m} \left(1 - \frac{P_{\Phi 0}^2}{8c^2m^2R_0'^2} - \frac{G(2\gamma+1)M}{c^2R_0'} \right) - \frac{Gm^2}{r} \left(1 - \frac{P_{\Phi 0}^2}{8c^2m^2R_0'^2} - \frac{2(2\beta-1)GM}{c^2R_0'} \right) - \\
& - \frac{GmMr^2}{2R_0'^3} P_2 \left(\frac{x \cos(n't) + y \sin(n't)}{r} \right) - \frac{P_{\Phi 0}^2}{4c^2m^3R_0'^2} (-p_x \sin(n't) + p_y \cos(n't))^2 - \\
& - \frac{G(2\gamma+1)P_{\Phi 0}'Mr}{c^2mR_0'^3} (-p_x \sin(n't) + p_y \cos(n't)) + \\
& + \frac{GP_{\Phi 0}'^2}{8c^2r^3R_0'^2} (-x \sin(n't) + y \cos(n't))^2 + O\left(10^{-12}\frac{p_r^2}{m}\right).
\end{aligned} \tag{11}$$

The relativistic terms retained are of the order of $O(10^{-8}p_r^2/m)$. Passing to the comoving coordinate system (which rotates such that the axis Ox joints the massive body with the barycenter of the binary system), we obtain the autonomous Hamiltonian:

$$\begin{aligned}
H_{bin} = & \frac{p_r^2}{m} \left(1 - \frac{P_{\Phi 0}^2}{8c^2m^2R_0'^2} - \frac{G(2\gamma+1)M}{c^2R_0'} \right) - \frac{Gm^2}{r} \left(1 - \frac{P_{\Phi 0}^2}{8c^2m^2R_0'^2} - \frac{2(2\beta-1)GM}{c^2R_0'} \right) + \\
& + n' (xp_y - yp_x) - \frac{GmM}{4R_0'^3} (3x^2 - r^2) - \frac{P_{\Phi 0}^2}{4c^2m^3R_0'^2} p_y^2 + \\
& - \frac{G(2\gamma+1)P_{\Phi 0}'M}{c^2mR_0'^3} r p_y + \frac{GP_{\Phi 0}'^2}{8c^2R_0'^2} \frac{y^2}{r^3} + O\left(10^{-12}\frac{p_r^2}{m}\right).
\end{aligned} \tag{12}$$

The influence of the PPN model of the 1+2 problem is:

- a) a direct one due to the presence of the last three terms in Eq. (12),
- b) an indirect one, by the modification of the coefficients which appear in the other terms, also present in the Newtonian case.

4 CONCLUSIONS

The main purpose of this paper is to present a theoretical problem which may constitute the premise of a GRT experiment. The prelimined development of Solar System space missions could create the technological premises for setting up such a 1+2 body system, formed of equal masses, on a precise given orbit, with the purpose to "detect" the influence of some PPN parameters in order

to evaluate the GRT predictions. To become real, this model should take into account facts as: the presence of other bodies, the shape of the massive body, a more realistic orbit of the barycenter than the circular one, the radiation pressure, the influence of the electromagnetic field, etc.

The study of this type of problem has some advantages:

- the possibility of choosing the appropriate initial conditions: the masses rapport, the semi-major axis of the binary system mass center motion relative to the massive body, the orbital elements of the relative motion of the two-body system related to the orbital plane of the mass center motion;

- the "faster evolution of the time" in the sense that one needs a shorter interval of time for performing the experiment than in the case of the study of a natural body motion;

- the possibility of realizing more precise measurements for the obtained effects, due to the fact that the bodies in the binary system are very close and due to the high precision given by the present measuring devices which may be set up on-board.

One notices here that, for the chosen initial data, in order to obtain the relativistic correction for the indirect effects, it is necessary a first order theory with the small parameter λ_1 . In order to obtain the direct effects a second order theory should be performed by means of standard procedures.

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